WESLEY COLLEGE
By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER TWO 2019
TEST 5: Differentiation and Differential Equations

Name: Master

Friday 30th August 2019

Time: 55 minutes

Total marks: $\frac{}{20} + \frac{}{30} = \frac{}{50}$

Calculator free section – maximum 25 minutes

1. [~~5~~ 6 marks]

An electrical device, subject to a constant voltage of 24 Volts, has a resistance R that is decreasing at a rate of 0.1 Ohm per second.

(An Ohm is the standard unit of electrical resistance.)

The voltage V, current I (in Ampere) and resistance R follow Ohm's Law: $V = I \times R$

Describe (quantitatively) how the current is changing when the resistance is 4.0 Ohm.

$\frac{dR}{dt} = -0.1$ ✓

$V = IR \Rightarrow \frac{dV}{dt} = I \cdot \frac{dR}{dt} + \frac{dI}{dt} \cdot R$ ✓✓

$0 = 6(-0.1) + \frac{dI}{dt} \cdot 4$ ✓

$\frac{dI}{dt} = \frac{0.6}{4} = 0.15$ ✓

i.e. current is increasing at a rate of 0.15 Amp/sec. ✓

or $I = \frac{V}{R}$

$\therefore \frac{dI}{dt} = \frac{V \cdot (-1)}{R^2} \frac{dR}{dt} = \frac{24 \times 0.1}{16} = 0.15$

2. [9 marks – 2, 4 and 3]

(a) A particle is travelling in a straight line with velocity v related to displacement x by the equation: $v = 2\sqrt{x-1}$. Show that acceleration a is a constant

Either $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ ✱

$= \frac{d}{dx}(2x-2)$ ✓

$= 2$ ✓, a constant

or $a = v.\frac{dv}{dx}$ ✓

$= 2\sqrt{x-1}\,.\,2.\frac{1}{2}(x-1)^{-\frac{1}{2}}$

$= 2$ ✓

(b) For $v = 2\sqrt{x-1}$, determine x as a function of time t, if $x(t=0) = 5$

$\frac{dx}{dt} = 2\sqrt{x-1}$

$\int \frac{dx}{\sqrt{x-1}} = \int 2\,dt$ ✓

$2\sqrt{x-1} = 2t + c$ ✓

$(0,5) \Rightarrow 4 = c$ ✓

$\therefore \sqrt{x-1} = t+2$

$x-1 = (t+2)^2$

$x = (t+2)^2+1$ ✓ $= t^2+4t+5$

or $\frac{dv}{dt} = 2$

$\Rightarrow \int dv = \int 2\,dt$

$\Rightarrow v = 2t + c$

$2\sqrt{x-1} = 2t + c$

$(0,5) \Rightarrow c = 4$

& etc ...

(c) If acceleration $a = \cos x$, find v in terms of x when $v\left(x = \frac{\pi}{2}\right) =$ 2 and $v \geq 0$.

$\cos x = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\therefore \int \cos x = \frac{1}{2}v^2$ ✓

$\sin x + c = \frac{1}{2}v^2$

$\left(\frac{\pi}{2}, 2\right) \Rightarrow 1 + c = 2$ ✓, $c = +1$

$\therefore v^2 = 2\sin x + 2$

$\therefore v = \sqrt{2\sin x + 2}$, $v \geq 0$

✓

3. [5 marks – 4 and 1]

A population of bacteria, P at time t, is growing at a rate modelled by:

$$\frac{dP}{dt} = P - \frac{P^2}{1000}$$

(a) Show, by differentiation (and substitution), that $P = \dfrac{1000}{1+Ce^{-t}}$ satisfies this differential equation, for any value of the constant C.

$$\frac{dP}{dt} = 1000 \times (1+Ce^{-t})^{-2} \times (-1) \times (-1) \times Ce^{-t} = \frac{1000\, Ce^{-t}}{(1+Ce^{-t})^2}$$

$$P = \frac{1000}{1+Ce^{-t}} \Rightarrow Ce^{-t} = \frac{1000}{P} - 1$$

$$\therefore \frac{dP}{dt} = \frac{\left(\frac{1000}{P} - 1\right) \times 1000}{\left(\frac{1000}{P}\right)^2}$$

$$= \left(\frac{1000}{P} - 1\right) \times \frac{P^2}{1000^2} \times 1000$$

$$= P - \frac{P^2}{1000}$$

(b) Calculate C if $P(t=0) = 10$

$$10 = \frac{1000}{1+C} \Rightarrow 1+C = 100$$

$$C = 99$$

Year 12 Specialist Test 5: Derivatives and Differential Equations

Name: ______________________

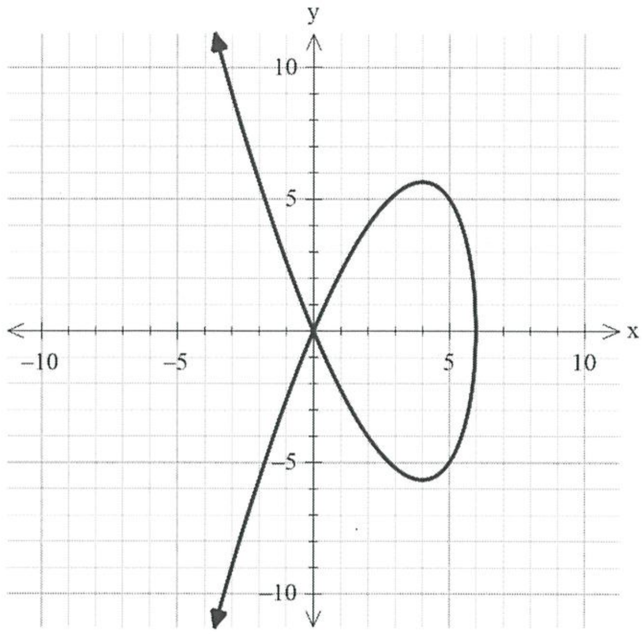
Time: 35 minutes

30 marks

Calculator assumed section

4. [7 marks – 2, 2 and ~~2~~ 3]

The curve defined by $y^2 = x^2(6-x)$, as shown, is another right strophoid.



(a) Derive an expression for $\frac{dy}{dx}$ in terms of both x and y.

$2y\frac{dy}{dx} = 12x - 3x^2$ ✓

$\frac{dy}{dx} = \frac{12x - 3x^2}{2y}$ ✓

(b) Determine the exact co-ordinates of the relative minimum and maximum points on the closed part of the curve.

$12x - 3x^2 = 0 \Rightarrow x = 0$ or 4

$x = 4$ defines min / max points ✓

Max at $(4, 4\sqrt{2})$; min at $(4, -4\sqrt{2})$ ✓

(c) Investigate the value(s) of the slope of the curve at the origin. The graph shows that these slopes are defined!

$\frac{dy}{dx} = \frac{12x - 3x^2}{\pm 2x\sqrt{6-x}}$ ✓ $= \pm\frac{12 - 3x}{2\sqrt{6-x}} = \pm\frac{12}{2\sqrt{6}}$ ✓ at $x = 0$

$\Rightarrow$ slopes are $\pm\sqrt{6}$ ✓

OR $y = \pm x\sqrt{6-x}$

$\Rightarrow \frac{dy}{dx} = \pm\left(\sqrt{6-x} - \frac{x}{2\sqrt{6-x}}\right)$ and $\frac{dy}{dx}\Big|_{x=0} = \pm\sqrt{6}$.

5. [10 marks – 1, 2, 2, 2, 1, 1 and 1]

The individual seat bookings, B, for a school production are increasing at a rate modelled by

$$\frac{dB}{dt} = kB(3800 - B)$$

(a) What is the maximum number who might attend this production?

✓ 3800

At the instant when 80 bookings had been made, bookings were increasing at a rate of 50 per day.

(b) Show clearly that $k = \dfrac{1}{5952}$

$50 = k \times 80 \times 3720$ ✓

$k = \dfrac{50}{80 \times 3720}$

$= \dfrac{1}{8 \times 744}$ ✓ $= \dfrac{1}{5952}$ or ClassPad.

(c) What is the maximum rate of increase of bookings?

$\frac{dB}{dt}$ is a parabola with max value at T.P.

TP at $B = 1900$ ✓

$\left.\frac{dB}{dt}\right|_{1900} = \frac{1}{5952} \cdot 1900^2 = 606.52$ ✓

i.e. 606 or 607 bookings per day.

After three days, 256 seats had been booked.

(d) Write an equation to represent the number of bookings as a function of t.

$P = \dfrac{3800 \times 40}{40 + 3760\, e^{-\frac{3800}{5952}t}}$ ✓✓ or similar $(P_0 = 40)$

$= \dfrac{3800}{1 + \frac{3760}{40} e^{-\frac{3800}{5952}t}}$

$= \dfrac{3800}{1 + 94\, e^{-0.63844T}}$

Question 5 (continued)

Determine the:

(e) initial number of bookings

$P_0 = 40$ ✓

(f) number of bookings made in the first 8 days

$P(8) = 2422$ ✓

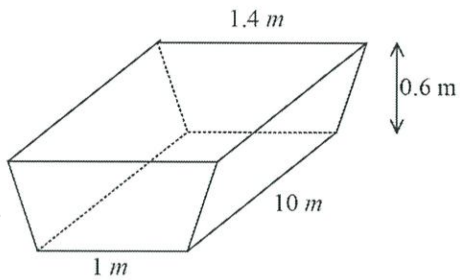
(g) day on which bookings close because only 100 seats remain unsold.

$P = 3700 \Rightarrow t = 12.77$

i.e. on the 13^{th} day. ✓

6. [4 marks]

A water trough 10 m long has a trapezoidal cross-section as shown.



It is being filled at a rate of 60 litres per minute.

When the water is h m deep, the volume V (in m^3) is given by $V = 10\left(h + \frac{h^2}{3}\right)$

How fast is the water level rising:

(a) initially (when $h = 0$)

(b) when $h = 0.3$ m

$$\frac{dV}{dt} = 10\frac{dh}{dt} + \frac{20h}{3}\frac{dh}{dt} = 60 \text{ L/min} = 0.06 \text{ m}^3\text{/min}$$ ✓ ✓

(a) $h = 0 \Rightarrow \frac{dh}{dt} = \frac{1}{10} \times 0.06 = 0.006$ m/min

or 6 mm/min ✓

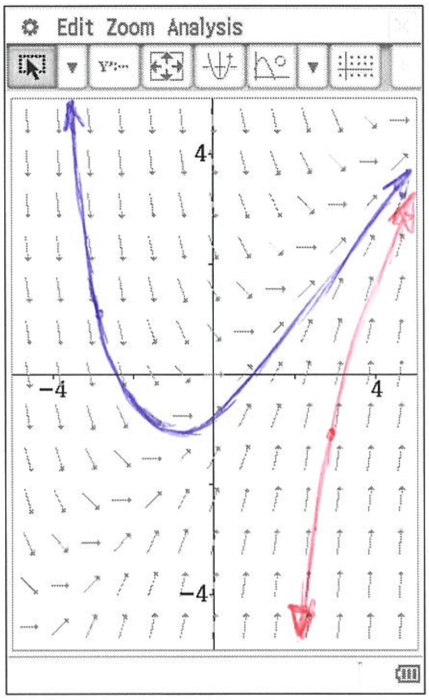
(b) $h = 0.3 \Rightarrow \frac{dh}{dt} = \frac{0.06}{10 + 2} = 0.005$ m/min

or 5 mm/min ✓

7. [9 marks – 1, 2, 1, 2 and 3]

Slope or gradient fields enable us to analyse differential equations that are difficult to solve.

Consider $\frac{dy}{dx} = x - y$, as shown.



(a) Describe the locus of points with a horizontal gradient.

the line $y = x$

✓

(b) Sketch the solution to $\frac{dy}{dx} = x - y$ that passes through $(-3,1)$

✓✓

(c) Sketch the solution to $\frac{dy}{dx} = x - y$ that passes through $(3,-1)$

✓

(d) Describe and generalise the differences between these solutions in (b) and (c)

✓ Different behaviours depending on the given point (boundary condition). Above $y = x$, has TP & asymptote
✓ Below $y = x$ has no TP and 1 asymptote.

(e) Use Euler's method, with $\delta x = 0.1$ to estimate $y(x = 2.4)$ for the solution that passes through the point $(2,2)$

Use calculator:

x	y	δx	δy
2	2	0.1	0
2.1	2	0.1	0.01
2.2	2.01	0.1	0.019
2.3	2.029	0.1	0.0271
2.4	2.0561		

✓ ✓

✓